Numerical Studies: Adaptive Mixed Finite Element Schemes for Benjamin-Bona-Mahony and Burgers Equations

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ABSTRACT

Adaptive mixed finite element schemes for Benjamin-Bona-Mahony and Burgers equations are considered. The aim is to generate more accurate approximate solutions of the exact solution and its derivative by using adaptive mesh refinement scheme. The mesh is refined by using two proposed methods with selected tolerance to achieve approximate solutions having specified accuracy in an optimal way. The computation begins with an experimental set of approximate solutions generated on a coarse initial mesh by using the mixed finite element method. Then, the error estimate of these solutions is appraised. A principal tool of this adaptive scheme is the availability of local error estimates, namely local a posteriori error estimators. The numerical results show that the desired accuracy of approximate solution can be accomplished by using the proposed adaptive schemes.

Keywords: *Adaptive schemes, a posteriori, BenBM equation, Burgers equation, mixed finite element method.*

I. INTRODUCTION

Adaptive schemes of finite element methods for numerical solutions of partial differential equations are considered as a standard tool in science and engineering to achieve better accuracy with minimum degrees of freedom. The adaptive scheme for one dimensional boundary value problems is studied in 1984 by Babuska et.al [3]. Studies about adaptive schemes with mixed finite element methods have been carried out in [4]-[7], [10] and the references therein. Details on the theory of adaptive finite element methods can be referred to [8].

In this paper, we study adaptive schemes of mixed finite element method (MFEM) for two nonlinear time dependent partial differential equations, namely the Benjamin-Bona-Mahony (BenBM) equation represented by

$$\partial_t u - \partial_{xxt} u + \partial_x u(u+1) = 0\,, \tag{1}$$

and the Burgers equation

$$\partial_t u - \frac{1}{2}\partial_{xx} u + u\partial_x u = 0 \tag{2}$$

for $x \in \Omega,\ t \in (0,T]$. The Dirichlet boundary conditions for (1) and (2) are defined by

$$\begin{aligned} &u(0,t) = 0, \quad t \in [0,T] \\ &u(1,t) = 0, \quad t \in [0,T] \end{aligned} \tag{3}$$

and initial condition of both equations as

$u(x,0) = u_0(x),$

for $x \in \Omega$. We note that the notations in (1) and (2) are representing the following definitions

$$\partial_t u = \frac{\partial u(x,t)}{\partial t}, \partial_x u = \frac{\partial u(x,t)}{\partial x},$$

$$\partial_{xx} u = \frac{\partial^2 u(x,t)}{\partial x^2}, \partial_{xxt} u = \frac{\partial^3 u(x,t)}{\partial x^2 \partial t},$$

T is a positive constant and Ω is the interval $(0,1)$.

The aim is to generate more accurate approximate solutions of the exact solution $u(x,t)$ by implementing adaptive mesh refinement scheme for both equations. Using the adaptive schemes of MFEM, we try to refine a mesh to achieve approximate solutions having specified accuracy in an optimal way. The computation begins with an experimental set of approximate solutions generated on a coarse initial mesh by using the MFEM. The MFEM implemented in this study is based on an approach for nonlinear parabolic equations suggested by Pani, the H^1–Mixed Finite Element Method (H^1MFEM) [9]. The method permits the approximation for the exact solution of the problem, u and the derivative of u. Then, the error estimate of these solutions is appraised. This is due to a principal tool of this adaptive scheme, which is the accessibility of local (element wise) error estimates, namely local a posteriori error estimators (AP$_{EE}$) [1]-[2]. The details of AP$_{EE}$ where the approximation of the exact solutions are computed using the H^{1}MFEM for the BenBM equation and the Burgers equation are presented in [12] and [11], respectively. Using the results obtained in both studies, we experiment the error estimates. If the error estimate fails to satisfy the prescribed accuracy, refinement are made to obtain the desired approximate solutions with minimum effort..

II. ADAPTIVE SCHEMES

The basic idea of the adaptive schemes is to create a refined partition (mesh) with the given of approximate solution, by partitioning elements where the error estimators indicate that the errors are large.

Then, on this refined partition, the next approximate solution is computed. The process is repeated until the desired accuracy of approximated solution is obtained. The final outcomes of the adaptive scheme are the refined mesh and accurate approximate solutions.

A principal tool for this adaptive scheme is the availability of local error estimates, namely local AP$_{EE}$. An adaptive scheme of H^{1}MFEM consists of successive loops of four steps as shows in Figure 1.

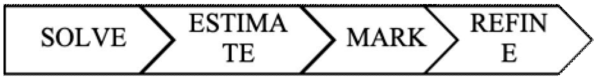

Figure 1: Four steps of an adaptive scheme

The step SOLVE involves the computation of the approximate solution by using H^{1}MFEM. Secondly, in step ESTIMATE, AP$_{EE}$ are computed locally on each element. The computation of approximate solutions and local a posteriori error estimations for the BenBM and Burgers equations are respectively presented in [12] and [11]. Some of the equations are restated to smoother the explanation of these adaptive schemes.

The AP$_{EE}$ are then evaluated in MARK step to appraise the accuracy of the approximate solution and to control the adaptive enrichment through the refinement process. Given a tolerance κ , we check the global AP$_{EE}$. If the global AP$_{EE}$ is larger than κ , then we appraise ω largest local AP$_{EE}$, $\omega = 1, 2, 3, \cdots$ where ω is decided by the user. The elements of these respective local AP$_{EE}$ are identified for a mesh refinement process in step REFINE. The adaptive strategy applied in REFINE is based on halving the ω intervals where the local AP$_{EE}$ are the largest.

In the refinement process of step REFINE, we introduce an Adaptive Method, which is implemented in two different ways in this study. After the refinement process, a new mesh is generated. By using this finer mesh, the process is repeated and is terminated when the global error estimator is smaller than the assigned tolerance κ .

III. ADAPTIVE PROCEDURES

In the section, we referring to [12] and [11] for the notations, definitions and derivation of the related equations. Some of the notations, definitions and derivation are restated to smoother the elaboration of findings of this study.

For $a = 0,1,\ldots$ let $\langle .,. \rangle_a$ represents the inner product and let $\|.\|_{H^a(Z)}$ represents a norm in any normed of vector space Z. Next, we let $H^0(\Omega)$ be the Lebesgue space $L^2(\Omega)$.

We also note that the $H^1(Z)$ represents the Sobolev space consists of functions m for which the norm is represented by

$$\|m\|_{H^1(\Omega)} = \sqrt{\|m\|^2_{H^0(\Omega)} + \|\partial_x m\|^2_{H^0(\Omega)}} .$$

The space $H_0^1(Z)$ is a vector space which includes all functions in $H^1(Z)$ with zero value at the endpoints of Ω . We use the inner product notation as $\langle u,v \rangle_a$ instead of $\langle u(t),v \rangle_a$ to reduce the packed notations, when there is no misunderstandings. For $j>0$ and $u,v \in H^s(Z_j)$ we let

$$\langle u,v \rangle_{s,z_j} = \int_{Z_j} u(x)v(x)dx$$

be the local inner product in $H^s(Z_j)$. The interval Ω is partitioned into

$$0 = X_1 < X_2 < \ldots < X_N < X_{N+1} = 1$$

and let $h_j := X_{j+1} - X_j,\ j = 1,\ldots,N$, and h represents the maximum value of h_j .

Similar to [12] and [11], the hat functions ψ_{l1} on (X_{j-1}, X_{j+1}) for $j = 2,\ldots,N$, is used to define the linear basis functions, that is

$$\psi_{l1}(x) = \begin{cases} \dfrac{x - X_{j-1}}{h_{j-1}}, & x \in [X_{j-1}, X_j) \\ \dfrac{X_{j+1} - x}{h_j}, & x \in [X_j, X_{j+1}) \\ 0, & \text{otherwise} \end{cases} \qquad (4)$$

At the endpoints of Ω we define

$$\psi_{11}(x)=\begin{cases}\dfrac{X_2-x}{h_1}, & x\in[X_1,X_2)\\ 0, & \text{otherwise}\end{cases}\tag{5}$$

and

$$\psi_{N+1,1}(x)=\begin{cases}\dfrac{x-X_N}{h_N}, & x\in[X_N,X_{N+1})\\ 0, & \text{otherwise}\end{cases}\tag{6}$$

For $j=1,\ldots,N$ and $k=2,3,4,\ldots$, let functions ψ_{jk} be defined as antiderivative of the Legendre polynomials, which are scaled to the subinterval $[X_j,X_{j+1}]$. The derivation of ψ_{jk} is represented by Eq. (13) in [12] and Eq.11 in [11].

Let S_h represents the space of piecewise linear functions on Ω. Then, let $\mathring{S}_h$ be the subspace of S_h consisting of functions vanishing at the endpoints. Let S_h^k be the space that span $\psi_{1k},\ldots,\psi_{N,k}$ at $k=2,3,4,\ldots$.

In addition, following the notation and derivation in [12] and [11], let V_h^p be the finite dimensional subspace of $H^1(\Omega)$ and $\mathring{V}_h^p$ be the finite dimensional subspace of $H_0^1(\Omega)$.

A. Step SOLVE of adaptive scheme

Firstly, under the step SOLVE, we compute the approximate solutions U_h and V_h of both equations, where V_h is the approximation of derivative of u. By referring to [12], values of approximate solution for the BenBM equation are computed by using the Galerkin equations of the problem. By using the H^1MFEM, (1) and (3) are approximated by semidiscrete approximation, namely

$$\langle\partial_x U_h,\partial_x\chi_h\rangle_0=\langle V_h,\partial_x\chi_h\rangle_0 \qquad \forall\chi_h\in\mathring{V}_h^p,\ \ t\in(0,T],\tag{8}$$

$$\langle\partial_t V_h,w_h\rangle_1=\langle U_hV_h,\partial_x w_h\rangle_0+\langle V_h,\partial_x w_h\rangle_0 \qquad \forall w_h\in V_h^q,\ \ t\in(0,T],\tag{9}$$

and

$$\langle V_h(0),w_h\rangle_1=\langle v_0,w_h\rangle_1 \quad \forall w_h\in V_h^q, t=0.\tag{10}$$

The matrix representations of (8)-(10) are represented as

$$S_pU(t)=B_{p,q}V(t),\tag{11}$$

and

$$(M_q+S_q)\partial_tV(t)=H\left(U(t),V(t)\right).\tag{12}$$

We note that the $S_p,B_{p,q},M_q$ and S_q are matrices defined in [12] and the vector $H\left(U,V\right)$ is an $(Nq+1)\times 1$ vector, where

$$H^{(0)}=\begin{bmatrix}\left\langle(U+1)V,\psi_{1,1}\right\rangle_0\\ \vdots\\ \left\langle(U+1)V,\psi_{N+1,1}\right\rangle_0\end{bmatrix}$$

and

$$H^{(j)}=\begin{bmatrix}\left\langle(U+1)V,\psi_{j,2}\right\rangle_0\\ \vdots\\ \left\langle(U+1)V,\psi_{j,q}\right\rangle_0\end{bmatrix}$$

for $j=1,\ldots,N$.

Similarly, by referring to [11], the approximate solutions U_h and V_h for the Burgers equation is computed by solving (8) and

$$\langle\partial_tV_h,w_h\rangle_0+\frac{1}{2}\langle\partial_xV_h,\partial_xw_h\rangle_0=\langle U_hV_h,\partial_xw_h\rangle_0 \qquad \forall w_h\in V_h^q,\ \ t\in(0,T],\tag{13}$$

and

$$\langle V_h(0),w_h\rangle_0=\langle v_0,w_h\rangle_0 \quad \forall w_h\in V_h^q,t=0.\tag{14}$$

The matrix representations of (8), (13)-(14) are represented as

$$S_pU(t)=B_{p,q}V(t),$$

$$M_q\partial_tV(t)+\frac{1}{2}S_qV(t)=G\left(U(t),V(t)\right).$$

Noting that $S_p,B_{p,q},M_q$ and S_q are matrices defined in [11] and the vector $G\left(U(t),V(t)\right)$ is an $(Nq+1)\times 1$ vector, that is

$$G^{(0)}=\begin{bmatrix}\left\langle UV,\psi_{1,1}\right\rangle_0\\ \vdots\\ \left\langle UV,\psi_{N+1,1}\right\rangle_0\end{bmatrix}$$

and

$$G^{(j)}=\begin{bmatrix}\left\langle UV,\psi_{j,2}\right\rangle_0\\ \vdots\\ \left\langle UV,\psi_{j,q}\right\rangle_0\end{bmatrix}$$

for $j=1,\ldots,N$.

B. Step ESTIMATE of adaptive scheme

We compute the local AP$_{\mathrm{EE}}$ E_j and F_j for $j=1,\dots,N$ by using the linear parabolic error estimate method, introduced in [12] and [11]. The AP$_{\mathrm{EE}}$ E and F locally on each subinterval Ω_j of the BenBM equation are computed by using Eq. 24 - Eq. 26 stated in [12] and Eq. 23 - Eq. 24 stated in [11].

Then, the computation of AP$_{\mathrm{EE}}$ E and F on element j for the BenBM equation is done by noting that

$$E(x,t)=\sum_{j=1}^{N} E_j(t)\psi_{j,p+1}(x),$$

$$F(x,t)=\sum_{j=1}^{N} F_j(t)\psi_{j,q+1}(x),$$

and by using

$$\begin{aligned}&\langle \partial_t F(t),\psi_{j,q+1}\rangle_{1,\Omega j}-\langle U_h(t)F(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega j}\\&-\langle V_h(t)E(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega_j}-\langle F(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega_j}\\&=\langle U_h(t)V_h(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega j}+\langle V_h(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega_j}\\&-\langle \partial_t V_h(t),\psi_{j,q+1}\rangle_{1,\Omega j}\end{aligned}\tag{15}$$

and

$$\begin{aligned}&\langle \partial_x E(t),\partial_x\psi_{j,p+1}\rangle_{0,\Omega j}\\&=\langle F(t),\partial_x\psi_{j,p+1}\rangle_{0,\Omega j}+\langle V_h(t),\partial_x\psi_{j,p+1}\rangle_{0,\Omega j}.\end{aligned}\tag{16}$$

Similarly, for the Burgers equation, as stated in [11], the E_j and F_j are computed by using (16) and

$$\begin{aligned}&\langle \partial_t F(t),\psi_{j,q+1}\rangle_{1,\Omega j}+\frac{1}{2}\langle \partial_x F(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega j}\\&-\langle U_h(t)F(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega_j}\\&-\langle V_h(t)E(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega_j}\\&-\langle F(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega_j}\\&=-\frac{1}{2}\langle \partial_x V_h(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega j}\\&+\langle U_h(t)V_h(t),\partial_x\psi_{j,q+1}\rangle_{0,\Omega j}\\&-\langle \partial_t V_h(t),\psi_{j,q+1}\rangle_{1,\Omega_j}.\end{aligned}\tag{17}$$

C. Step MARK of adaptive scheme

Using the computed local AP$_{\mathrm{EE}}$, the global AP$_{\mathrm{EE}}$ E and F which are the summation of the local AP$_{\mathrm{EE}}$ E_j and F_j for $j=1,\dots,N$ is appraised with the assigned tolerances κ_U and κ_V such that

$$\|E(t)\|_{H^1}\le\kappa_U$$

and

$$\|F(t)\|_{H^1}\le\kappa_V.$$

The process is stopped if the conditions are satisfied. The ω largest local AP$_{\mathrm{EE}}$ E_j are identified for marking process.

D. Step REFINE of adaptive scheme

In this step, we test the problems by using Adaptive Method (AM). The AM is applied in two different ways, namely AM[1] and AM[3]. The AM[1] is referring to cases with $\omega=1$, which is the refinement is applied only on the largest AP$_{\mathrm{EE}}$ E_j for $j=1,\dots,N$. While AM[3] with $\omega=3$ referring to refinement process on three largest AP$_{\mathrm{EE}}$ E_j for $j=1,\dots,N$.

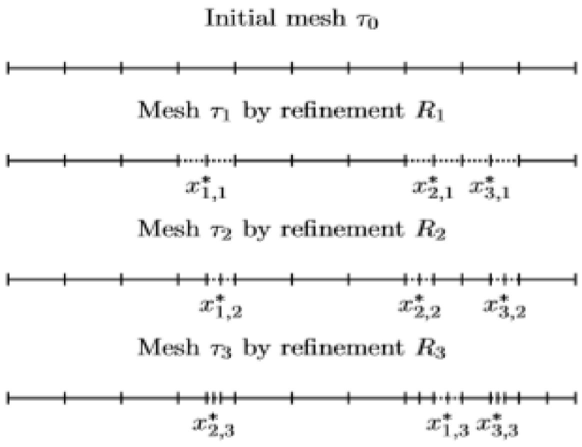

Figure 2: Refinement processes by AM[3].

Then, the new mesh generated from the step MARK and REFINE is used to generate the new approximate solution in the step SOLVE. The process is repeated and stopped when the desired accuracy of the approximate solution is obtained.

IV. NUMERICAL RESULTS

The numerical experiments are computed using Matlab ODE solver and three methods, namely AM[1], AM[2] and non-adaptive method (NAM). The NAM is referring to cases with uniform meshes where all elements are refined in the refinement processes. As stated in Section II Adaptive Schemes, the global AP$_{\mathrm{EE}}$ is appraised according to the selected tolerance κ and the process is repeated and is terminated when the global error estimator is smaller than the assigned tolerance κ .

In the following numerical results, we noted that global AP$_{\mathrm{EE}}$ is computed for the exact solution of the problem, u and the derivative of u. Therefore, κ_U and κ_V are respectively selected as the tolerances for AP$_{EE}$ of u and the derivative of u. The tolerances are determined by using the information of hierarchical basis functions' degree (p and q) used to compute the approximate solutions. We also note that, the results are computed at $t=0.8$.

A. Numerical results of BenBM equation

Table I shows the numerical results of Example 1 computed with $\kappa_U = 5\times10^{2-p}$ and $\kappa_V = 5\times10^{1-q}$ for $p=2$ and $q=1$. The N represents the total number of elements generated by each method after the number of refinements. The number of refinements also represents the total number of problems required to solve, in order to get the approximate solution at the assigned tolerances. The Dof u and Dof v respectively represent the degree of freedom required to obtain the presented results.

Table I: Example 1 for the BenBM equation

	AM[1]	**AM[3]**	**NAM**
N	156	157	640
Dof u	311	313	1279
Dof v	157	158	641
Number of refinements	146	49	6
$\frac{\|E(t)\|_{H^1}}{\|u(t)\|_{H^1}}$	9.51E-04	7.90E-04	1.37E-04
$\frac{\|F(t)\|_{H^1}}{\|v(t)\|_{H^1}}$	2.24E-02	2.15E-02	1.49E-02

Based on the numerical results in Table I, AM[1] and AM[3] produced comparable error values with NAM. The NAM required $N=640$ while the AM[1] and AM[3] required N value less than 200. We note that, if the NAM is solved directly at $N=157$, the error values are larger, that is $\frac{\|E(t)\|_{H^1}}{\|u(t)\|_{H^1}} = 2.29\text{E-}03$,and $\frac{\|F(t)\|_{H^1}}{\|v(t)\|_{H^1}} = 6.08\text{E-}02$. In addition, Figure 3 shows the final refined meshes at $N=156$ by AM[1] and $N=157$ by AM[3].

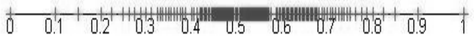

Figure 3(a): Refined meshes of Example 1 for the BenBM equation by AM[1]

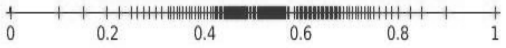

Figure 3(b): Refined meshes of Example 1 for the BenBM AM[3]

Table II: Example 2 for the BenBM equation

	AM[1]	**AM[3]**	**NAM**
N	44	46	160
Dof u	131	137	479
Dof v	89	93	321
Number of refinements	34	12	4
$\frac{\|E(t)\|_{H^1}}{\|u(t)\|_{H^1}}$	8.24E-04	2.83E-04	4.62E-05
$\frac{\|F(t)\|_{H^1}}{\|v(t)\|_{H^1}}$	6.42E-03	4.70E-03	2.03E-03

Table II shows the numerical results of Example 2 computed with $\kappa_U = 10^{3-p}$ and $\kappa_V = 10^{2-q}$ for $p=3$ and $q=2$. Figure 4 shows the final refined meshes by AM[1] and AM[3] for Example 2. The error values by NAM is slightly better than the error values by the AM[1] and AM[3]. However, higher degrees of freedom is required. We note that the error values by NAM (uniform refinement) at $N=46$, are larger, which are $\frac{\|E(t)\|_{H^1}}{\|u(t)\|_{H^1}} =$ 1.88E-03, and $\frac{\|F(t)\|_{H^1}}{\|v(t)\|_{H^1}} = 2.43\text{E-}02$.

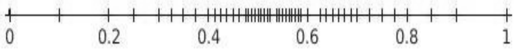

Figure 4(a): Refined meshes of Example 2 for the BenBM equation by AM[1]

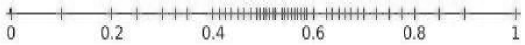

Figure 4(b): Refined meshes of Example 2 for the BenBM equation by AM[3]

B. Numerical results of Burgers equation

Table III shows the numerical results of Example 1 for the Burgers equation computed with $\kappa_U = 10^{-p-1}$ and $\kappa_V = 10^{-q-2}$ for $p = 2$ and $q = 1$, while Table IV shows the numerical results of Example 2 computed with $\kappa_U = 10^{-2p-2}$ and $\kappa_V = 10^{-2q}$ for $p = 3$ and $q = 2$.

Table III: Example 1 for the Burgers equation

	AM[1]	**AM[3]**	**NAM**
N	186	187	320
Dof u	371	373	639
Dof v	187	188	321
Number of refinements	176	59	5
$\frac{\|E(t)\|_{H^1}}{\|u(t)\|_{H^1}}$	8.14E-06	8.06E-06	4.14E-06
$\frac{\|F(t)\|_{H^1}}{\|v(t)\|_{H^1}}$	2.67E-03	2.66E-03	1.62E-03

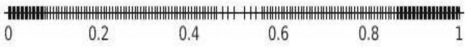

Figure 5(a): Refined meshes of Example 1 for the Burgers equation by AM[1]

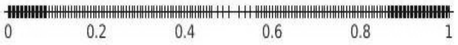

Figure 5(b): Refined meshes of Example 1 for the Burgers equation by AM[3]

Based on the results in Table III, the error values computed by the adaptive schemes are comparable with the result by the NAM and computed at lower number of elements N. Next, Figure 5 and Figure 6 respectively represent the final refined meshes by AM[1] and AM[3] for both examples of the Burgers equation.

Table IV: Example 2 for the Burgers equation

	AM[1]	**AM[3]**	**NAM**
N	95	97	160
Dof u	284	290	479
Dof v	191	195	321
Number of refinements	85	29	4
$\frac{\|E(t)\|_{H^1}}{\|u(t)\|_{H^1}}$	1.82E-08	1.76E-08	8.05E-09
$\frac{\|F(t)\|_{H^1}}{\|v(t)\|_{H^1}}$	9.55E-06	9.22E-06	8.33E-06

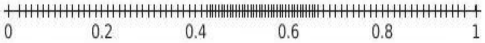

Figure 6(a): Refined meshes of Example 2 for the Burgers equation by AM[1]

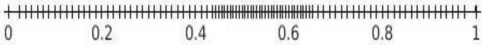

Figure 6(b): Refined meshes of Example 2 for the Burgers equation by AM[3]

V. CONCLUSION

In conclusion, based on the presented numerical results of adaptive schemes for the BenBM equation and the Burgers equation, we can see that the desired accuracy of approximate solution can be accomplished by using the proposed adaptive schemes. Besides, a comparable accuracy of approximate solution with lower degrees of freedom is obtained by these adaptive schemes. The marking process in MARK and refinement process in REFINE of adaptive procedure may be considered for the improvement and subject of further study.

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